Role of quenching on superdiffusive transport in two-dimensional random media

Pierre Barthelemy,[1,*] Jacopo Bertolotti,[1] Kevin Vynck,[1] Stefano Lepri,[2] and Diederik S. Wiersma[1]
[1]*LENS and CNR-INO, via Nello Carrara 1, 50019 Sesto Fiorentino, Florence, Italy*
[2]*Istituto dei Sistemi Complessi, CNR, via Madonna del Piano 10, 50019 Sesto Fiorentino, Florence, Italy*
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Transport in random media is known to be affected by quenched disorder. From the point of view of random walks, quenching induces correlations between steps that may alter the dynamical properties of the medium. This paper is intended to provide more insight into the role of quenched disorder on superdiffusive transport in two-dimensional random media. The systems under consideration are disordered materials called *Lévy glasses* that exhibit large spatial fluctuations in the density of scattering elements. We show that in an ideal Lévy glass the influence of quenching can be neglected, in the sense that transport follows to very good approximation that of a standard Lévy walk. We also show that, by changing sample parameters, quenching effects can be increased intentionally, thereby making it possible to investigate systematically diverse regimes of transport. In particular, we find that strong quenching induces local trapping effects which slow down superdiffusion and lead to a transient subdiffusivelike transport regime close to the truncation time of the system.

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I. INTRODUCTION

Various multiple scattering processes, such as light transport in common disordered materials, are known to obey standard diffusion dynamics [1]. The trajectories performed in such media are usually modeled as a Brownian random walk, assuming that steps are small and independent. It is well known, however, that significant deviations from these two assumptions may destroy the standard picture of diffusion and induce *anomalous diffusion phenomena* [2–4]. In particular, transport processes where the step length distribution has a diverging variance can no longer be described by regular Brownian motion but by the so-called Lévy walk processes [5–7].

Lévy walks are random walks performed at a constant velocity and whose step length distribution $P_L(l)$ is an α-stable Lévy distribution. These distributions have no simple expression except in a few specific cases (e.g., it is a Cauchy distribution when $\alpha=1$) [3]. They are generally defined through their Fourier transform as [8]

$$P_L(l)=\frac{1}{2\pi}\int_{-\infty}^{\infty}dk\, e^{-ikl}e^{-|ck|^{\alpha}}, \tag{1}$$

where the exponent α defines the decay of the step length distribution $[P_L(l)\sim 1/l^{\alpha+1}$, where $0<\alpha\leq 2]$ and the parameter c is a scaling parameter. Such distributions are said to be heavy tailed and are known to have a diverging variance for $\alpha<2$. Since transport is dominated by very long steps, the mean-square displacement for Lévy walks increases faster than linearly with time, hence the name superdiffusion. In particular, at large times t, we have

$$\langle x^2(t)\rangle \sim Dt^{\gamma}, \quad 1<\gamma<2, \tag{2}$$

where D is the generalized diffusion constant and γ is the superdiffusivity exponent, related to the α exponent as [9]

$$\gamma=\begin{cases}2, & 0<\alpha<1\\ 3-\alpha, & 1<\alpha<2.\end{cases} \tag{3}$$

Superdiffusion has been observed in several systems, including radiation transport in atomic vapors [10] and transport in turbulent flows [11]. Recently, some of us introduced a type of disordered optical material called a Lévy glass, in which light transport is superdiffusive [12]. Lévy glasses are realized by embedding transparent spheres of different sizes into a normal diffusive medium. These spheres create voids and thereby induce large spatial fluctuations of the scattering probability which modify locally the distances performed between successive scattering events. The sphere diameter probability distribution is set to decay as a power law to ensure a power-law decay of the step length distribution [13]. Owing to their ease of fabrication and tunability, Lévy glasses offer the opportunity to study anomalous transport processes in a systematic way.

One substantial aspect of transport in random media is the role played by quenched disorder. In quenched environments, the random-walk properties (allowed step length and directions, scattering probability, etc.) are determined by local quantities that do not evolve with time. Quenching involves step correlations, which may modify the dynamical transport properties of the random medium. For Lévy flights [i.e., random walks in which steps are distributed according to Eq. (1) and performed instantaneously], quenching has been shown to slow down superdiffusion [14–16], with the efficiency of this slowing down being determined by the dimensionality of the system [15,16]. In one-dimensional (1D) systems [17–20], quenched disorder has been shown to lead to significant corrections of the total transmission [18] and induce important differences between average transport properties and local ones [19]. While the influence of quenching is expected to decrease in higher dimensions, little is known about the behavior of two-dimensional (2D) and three-dimensional (3D) Lévy walks in systems with quenched disorder. Interestingly, the step lengths performed by a random walker in Lévy glasses are purely determined by its local environment. Transport in such systems therefore

*www.complexphotonics.org; barthelemy@lens.unifi.it

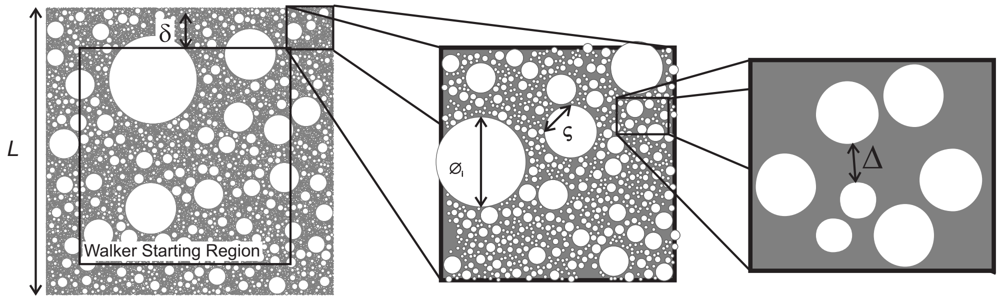

FIG. 1. Typical realization of a 2D Lévy glass, composed of a large set of transparent disks (in white) of diameter ϕ_i immersed in a scattering medium (in gray). The disk diameter distribution decays as a power law. Random walkers evolve within a square area of side length L, starting within a square area distant from the sample boundaries by $\delta = vT$. A chord length of a disk ς and the distance between two disks Δ are shown.

closely resembles a Lévy walk on quenched random media.

In this paper, we study the dynamical properties of 2D superdiffusive Lévy glasses by means of Monte Carlo simulations. The Lévy glasses are constituted of an assembly of disks immersed in a scattering region. Our main objective is to provide more insight into the role of quenching on superdiffusive transport processes in 2D random media. In Sec. II, we introduce the model considered throughout the paper, in particular the design parameters of 2D Lévy glasses and the Monte Carlo method. In Sec. III, we present a first-principles model for the step length distribution in 2D Lévy glasses and compare the results to those obtained from Monte Carlo simulations. We then provide evidence of superdiffusive transport by calculating the mean-square displacement of random walkers in the system. In Sec. IV, we show how the influence of quenching depends on the choice of sample parameters, allowing us to study different regimes of transport. A comparison is made of the various transport properties in the quenched and annealed cases, and the time evolution of the system is analyzed to times beyond the truncation time. It turns out that the asymptotic values of the diffusion constant at long times provide a measure for the effect of quenching in finite-size systems. Conclusions are presented in Sec. V.

II. MODEL

2D Lévy glasses were designed by placing randomly a large set of nonoverlapping disks of diameter varying by orders of magnitude in a large square area of side length L. A typical realization of a Lévy glass in such conditions is shown in Fig. 1. The disks were taken from $n=24$ different categories with diameters ϕ_i, geometrically spaced between $\phi_1=10$ and a maximum diameter of $\phi_n=14\,000$. The number of disks N_i of each diameter category was determined by the closest integer to

$$N_i = \frac{C}{\phi_i^{\beta+1}}, \tag{4}$$

where β describes the power-law decay of the disk diameter distribution. This exponent determines partly the decay of the step length distribution $P_L(l)$ in Lévy glasses [13] and is set to $\beta=1$. C is a proportionality constant given by

$$C = N_t \phi_1^{\beta+1} \frac{1-r}{1-r^n}, \tag{5}$$

where N_t is the total number of disks in the system and $r = (\phi_i/\phi_{i-1}) = 1.37$ is the ratio of the geometrical progression of the diameters. In practice, the discrete character of the diameter distribution should play no significant role on the transport properties, provided that the sampling is fine enough and follows the same rule [13]. Simulations with two different values of r ($r=1.37$ and $r=1.07$; $\phi_n=2000$) were performed and revealed no significant differences in the transport properties.

Knowing the disk diameter of each category and the corresponding number of disks, the sample size was determined by choosing a fixed disk filling fraction. The disks were placed randomly and uniformly in the sample area, from the largest to the smallest, checking after each placement that disks did not overlap each other. We used a disk filling fraction of $f=70\%$, which makes the sample realization relatively easy. Higher disk filling fractions are, in principle, possible; but the process of placing the spheres becomes more cumbersome. Once a high filling fraction is reached during the process of disk placing, the remaining space can be distributed in a way that prevents one to add the remaining disks. In this case the process has to be started again from zero. With a filling fraction of $f=70\%$, we never observed this issue.

Regarding the filling fraction a few comments are in place. In practice samples have a cutoff in maximum and minimum disk diameters. As a result of the latter it is impossible to reach $f=100\%$ filling fraction, whereas without cutoff it might be possible for certain configuration of the disks. Successive disk generations with decreasing diameters may fill the gaps left by the larger disks. Ideally, this process would be repeated *ad infinitum*, resulting in an interdisk region of zero measure (containing point scatterers). In practice, the cutoff in minimum disk size means that there is always a region of finite size in between the disks. A relevant parameter of the sample geometry is then the distance Δ between the boundaries of neighboring disks. This distance represents the thickness of scattering material a random walker has to cross to go from one disk to another. The average interdisk distance $\langle\Delta\rangle$ was estimated in our samples by averaging the shortest distances between 100 disks, chosen randomly, and their three respective nearest neighbors, yielding $\langle\Delta\rangle=9.82$. As shown below, the comparison between the mean free path in the scattering region and $\langle\Delta\rangle$ constitutes a good reference point to evaluate the importance of quenched disorder on transport. We will see that the effect of quenching becomes negligible when these two parameters are comparable, corresponding to the ideal case of having a single scattering event in between two voids (which would be obtained also in the ideal case of $f=100\%$ disk filling fraction with point scatterers in between the disks).

Monte Carlo simulations were performed to study the transport in these 2D Lévy glasses. Random walkers were asked to perform small steps of length $\delta_l=1$ at each time step of the simulation. The behavior of each random walker after each step was determined locally by the scattering probability p_s, equating zero within the disks, in which case the direction of the successive step would remain unchanged, and a nonzero value in the interdisk region, in which case a scattering event would possibly take place. To relate the strength of the scattering to the transport parameters, we defined the mean free path of the interdisk region as $\ell_s=-\delta_l/\ln(1-p_s)$. The random choice on a scattering event was performed at each step in the interdisk region by generating a random number $0\leq y<1$ and applying the following rule: if $y>p_s$, then the walker was not scattered; if $y<p_s$, a scattering event occurred: the new direction of the walker was chosen randomly according to a uniform distribution. The step length distribution $P_L(l)$ in a Lévy glass was evaluated by recording the distance between successive scattering events. The number of random walkers used to simulate transport in Lévy glasses was set to 10^5, after having checked that convergence was reached.

It is known that the initial conditions of a random-walk process are important in quenched systems [17]. In practical realizations of Lévy glasses, since the motion of random walkers always starts within the scattering region, considering an initial in-disk transport is unrealistic. The starting position of each random walker was therefore generated randomly within the interdisk region only. The initial position of the walkers was also bound to lie in a square area, slightly smaller than the sample area. The boundaries of this area were distant by $\delta=vT$ from the sample boundaries, where T is the maximum time of the simulation and v is the velocity of the walkers (set to 1 for the sake of simplicity). We also imposed $\delta<L/10$ in order to ensure that the starting positions of the random walkers would probe most possible local environments, and yet no walker could perceive the presence of the sample boundaries.

In order to apprehend the specific role played by quenched disorder, Monte Carlo simulations were also performed on their annealed counterpart. Annealed disorder implies that successive steps in a random-walk process are uncorrelated. The annealed systems were therefore modeled using the step length distributions $P_L(l)$ retrieved from Monte Carlo simulations in the quenched Lévy glasses. The inversion method [21] was used to calculate the random step length in the annealed system: the length l performed by a random walker at each step was generated according to $l=F^{-1}(y)$, where $F(x)$ is the cumulative distribution function $F(x)=\int_0^x P_L(l)dl$ and $0\leq y<1$ is a uniformly distributed random number.

III. TRANSPORT IN 2D LÉVY GLASSES

A. Step length distribution

The step length distribution $P_L(l)$ of Lévy glasses can be estimated from a first-principles model [13]. Consider a random walker penetrating a disk of diameter ϕ_i. Since the probability to be scattered within the disk is zero, the walker travels a distance ς along one of the chords of the disk before re-entering the scattering region. Only then it has a nonzero probability to be scattered and change direction. The probability for a walker to perform a distance $l-\varsigma$ after having crossed a disk decays exponentially, with a decay constant ℓ_s. Knowing the length of a chord ς, the step length probability can then be written as

$$P_{L|C}(l|\varsigma)=\frac{1}{\ell_s}e^{-(l-\varsigma)/\ell_s}H(l-\varsigma), \tag{6}$$

where $H(x)$ is the Heaviside function. The marginal step length distribution is then defined as

$$P_L(l)=\int_0^\infty P_{L|C}(l|\varsigma)P_C(\varsigma)d\varsigma. \tag{7}$$

Knowing the disk diameter ϕ_i, the chord length distribution $P_{C|D}(\varsigma|\phi_i)$ in a disk is given by [22]

$$P_{C|D}(\varsigma|\phi_i)=\frac{\varsigma}{\phi_i^2}\frac{1}{\sqrt{1-\varsigma^2/\phi_i^2}}. \tag{8}$$

The probability for a random walker to enter a disk of diameter ϕ_i in the Lévy glass is given by the product of the number of disks of diameter ϕ_i in the Lévy glass and the probability for a random walker to be in the neighborhood of such a disk, and is therefore proportional to the disk circumference. We then have

$$P_D(\phi_i)=AN_i\pi\phi_i\propto\frac{1}{\phi_i^{\beta}}, \tag{9}$$

where A is a normalization constant. The marginal chord length distribution is calculated as $P_C(\varsigma)=\Sigma_i P_{C|D}(\varsigma|\phi_i)P_D(\phi_i)$, and the step length distribution $P_L(l)$ is finally found by integrating Eq. (7).

Figure 2(a) shows the step length distribution in a 2D Lévy glass with a mean free path $\ell_s=10$ calculated with this approach and compared to that obtained from Monte Carlo simulations. A very good agreement is found between the two distributions, especially for long steps. Deviations can be explained by considering that the chord length model neglects the probability for walkers to cross more than one sphere in one step, as well as the probability to perform many consecutive steps in the scattering medium (recall that, in our case, the filling fraction of the scattering medium is $1-f=30\%$). These effects clearly depend on the ratio between the mean free path ℓ_s in the scattering region and the

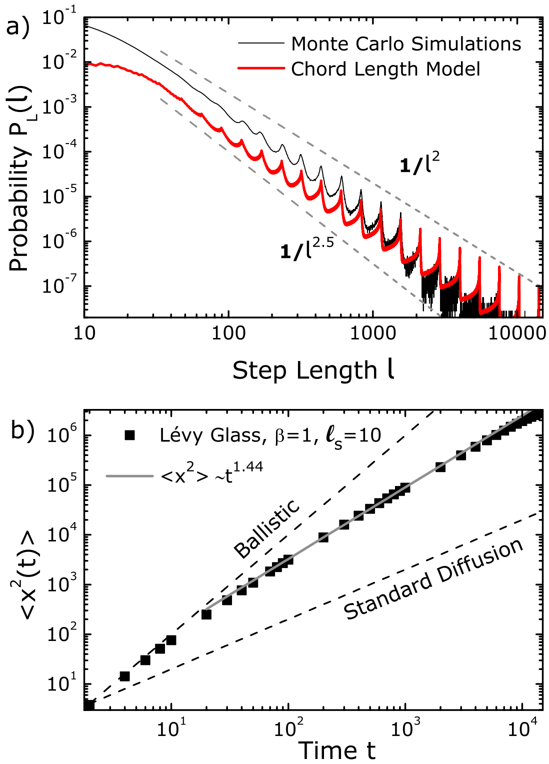

FIG. 2. (Color online) Superdiffusive transport in a 2D Lévy glass ($\ell_s = 10$, $\beta = 1$). (a) Step length distribution $P_L(l)$ obtained from Monte Carlo simulations (upper thin black curve) and from the chord length model (lower thick red curve). For large steps, the distribution displays strong oscillations and decays as a power law with an exponent comprised between 2 and 2.5. (b) Mean-square displacement $\langle x^2(t) \rangle$ calculated from Monte Carlo simulations (black squares) and from the standard Lévy walk model (solid gray line). The mean-square displacement is shown to increase faster than linearly with time, with an exponent $\gamma = 1.44$ that is consistent with a Lévy walk of exponent $\alpha = 1.56$. Deviations between Monte Carlo simulations and the standard Lévy walk model can be seen at long times.

average interdisk distance $\langle \Delta \rangle$. For short steps in particular, $P_L(l)$ is dominated by those performed within the interdisk region.

The chord length distribution in Eq. (8) presents a divergence for $\varsigma = \phi_i$. This divergence is characteristic of 2D systems and does not appear in three dimensions [13]. As this divergence is integrable, its influence on the step length distribution $P_L(l)$ is limited to the creation of finite-height peaks when $l = \phi_i$, thereby explaining the oscillations observed in Fig. 2(a). The amplitude of these oscillations increases for larger steps, which yields a certain difficulty to assign a precise value of α to the step length distribution. While the maxima of the oscillations decay closely to $1/l^2$, the minima decay slightly faster, approximately as $1/l^{2.5}$. In order to perform a comparison with the standard Lévy walk defined by Eq. (1), we divided the curve into a series of intervals delimited by the diameters of the disks and averaged the step lengths in each of them. The value of α was finally determined by fitting these averaged data with a power law decaying as $l^{-(\alpha+1)}$, yielding $\alpha = 1.56 \pm 0.04$. Note that the determination of α in 3D Lévy glasses is more straightforward, as the oscillations have a constant amplitude [13].

B. Superdiffusion

Due to the heavy tail of the step length distribution, Lévy glasses are expected to exhibit superdiffusive transport [12]. Figure 2(b) shows the mean-square displacement of random walkers in the Lévy glass considered above. For the very early times of the simulations ($t < \ell_s / v$), most walkers undergo ballistic transport. When they start being scattered in the disordered material, the mean-square displacement deviates from ballistic transport to enter a superdiffusive regime.

We now compare this behavior with the standard superdiffusion given by Eq. (3). The superdiffusion exponent we would expect for the calculated α exponent ($\alpha = 1.56$) is $\gamma = 1.44$. The agreement between Monte Carlo simulations and Eq. (3) is excellent at intermediate times ($t = 2 \times 10^1 - 4 \times 10^3$). Deviations from this superdiffusive transport can be observed at very long times. They can be explained by the finite maximum disk diameter, which imposes an upper truncation in the chord length distribution, leading in turn to a truncation of the step length distribution at $l \approx \phi_n$. Thus, for times longer than ϕ_n / v, transport is expected to become diffusive. (This point will be discussed in detail later on.) As in our case $\phi_n = 14\,000$, these effects remain weak in the simulation and therefore do not complicate the interpretation of the results.

IV. QUENCHED DISORDER

We now come to the main point of this paper, that is, to describe the role played by quenched disorder on superdiffusive transport in 2D Lévy glasses. The effect of quenching on transport can be isolated from other effects by comparing the transport properties of quenched Lévy glasses with those of the equivalent annealed systems. The procedure that we followed, as described in Sec. II, ensured that random walkers in the annealed media were subject to the exact same step length distribution as in the quenched Lévy glasses, and yet remained free of any correlation between successive steps.

In the first part of this section, we describe in detail how the influence of quenched disorder on superdiffusive transport in Lévy glasses can be controlled. We discuss the main features of transport due to quenching and explain their physical origin. The second part of this section is concerned with a study of transport in properly designed Lévy glasses in which quenching becomes negligible. The results of our simulations are compared with a standard Lévy walk model. The last part of this section is finally devoted to the effects of quenched disorder on the dynamical transport properties of Lévy glasses at their truncation time.

A. Role on transport

We chose to investigate the effects of quenched disorder on superdiffusive transport in 2D Lévy glasses by varying the mean free path ℓ_s in the interdisk region. Our motivation was that, as briefly discussed in Sec. II, the interdisk region is very likely to play a significant role on the transport prop-

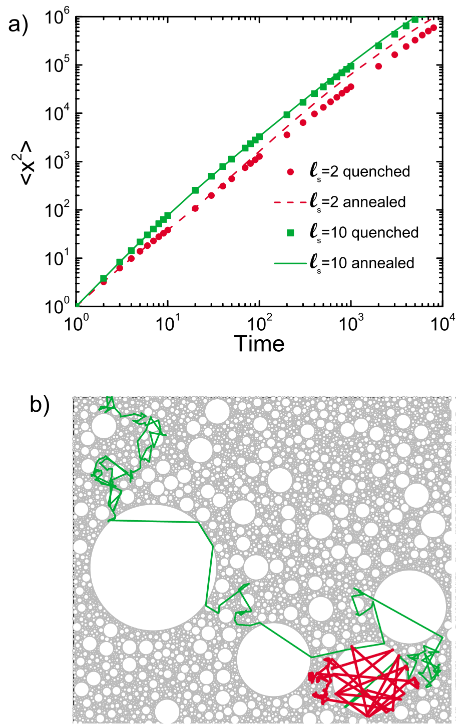

FIG. 3. (Color online) Influence of quenched disorder on transport. (a) Mean-square displacement versus time. For mean free path comparable to the interdisk distance ($\ell_s=10\approx\langle\Delta\rangle$), the mean-square displacement of the quenched system (green squares) is similar to annealed case (solid green curve). Instead, for mean free paths smaller than the interdisk distance ($\ell_s=2<\langle\Delta\rangle$), superdiffusion in the quenched system (red dots) is slowed down compared to the equivalent annealed system (dashed red curve). (b) Typical trajectory of random walkers in 2D Lévy glasses. For mean free paths much smaller then the interdisk spacing [$\ell_s=2<\langle\Delta\rangle$, red (dark gray) thick line], a random walker can be trapped in the neighborhood of a disk. The effect of the quenching on single random walkers is then extremely important. In a correctly designed Lévy glass the mean free path is equal or larger than the interdisk spacing [$\ell_s=10\approx\langle\Delta\rangle$, thin green (light gray) line] to avoid these trapping effects.

erties of Lévy glasses, in particular through the optical thickness $\langle\Delta\rangle/\ell_s$. This quantity represents the amount of scattering material a random walker has to cross in order to go from one disk to another. We effectively vary this quantity in our simulations by modifying the mean free path ℓ_s.

In Fig. 3(a), we compare the time evolution of the mean-square displacement in the quenched and annealed systems for two different values of ℓ_s at fixed interdisk spacing. As expected, the influence of quenching is found to depend strongly on the mean free path ℓ_s in the scattering medium. While for $\ell_s=10$ the difference in transport in the annealed and quenched systems is small, it becomes significant for $\ell_s=2$. This trend can be understood as follows. Consider a random walker performing a very large step of length l_1. In annealed systems, as there are no step length correlations, the following step, say l_2, will very likely be much smaller than l_1. The total displacement after two steps will then stay on the order of $\langle x^2\rangle\approx l_1^2$. In quenched structures, instead, after having crossed a large disk, any backscattering event will force the walker to re-enter the disk it just left. It is then more likely to perform a step of length $l_2\approx l_1$ in the opposite direction, thereby reducing the contribution of this walker to the mean-square displacement. In that sense, quenching may induce some *trapping* effects that force random walkers to stay in the vicinity of a given disk for a certain amount of time, with the backscattering probability depending on the ratio between the mean free path ℓ_s in the scattering medium and the average interdisk distance $\langle\Delta\rangle$. Such a trapping is clearly observed in Fig. 3(b). A random walker with a mean free path smaller than the average interdisk distance [thick red (black) trajectory, $\ell_s=2$] stays trapped in the neighboring region of a large disk for a certain amount of time. Its total displacement then does not exceed much the disk diameter. With a larger mean free path [thin green (gray) trajectory, $\ell_s=10$], the random walker can easily explore the whole sample and is less sensitive to the quenched nature of disorder. Note that this corresponds exactly to the optimum case discussed earlier in which on average one scattering event takes place in between the disks.

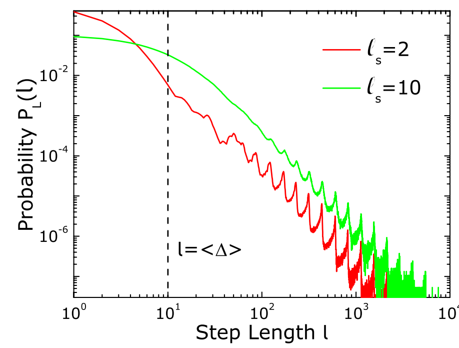

FIG. 4. (Color online) Step length distributions of 2D Lévy glasses for $\ell_s=2$ [red (dark gray) curve] and $\ell_s=10$ [green (light gray) curve]. For comparison, these distributions were also used to generate the step lengths of the corresponding annealed systems. For $\ell_s=2$, the distribution consists of a fast decaying part for $l<\langle\Delta\rangle$ and a more slowly decaying one for larger steps ($l>\langle\Delta\rangle$).

It is worth noting that in 1D systems the effect of quenching is much bigger since any backscattering event forces the walker to go back exactly on its tracks, thereby completely canceling its contribution to the two-step mean-square displacement. In 2D systems, an event completely canceling the contribution of a step is very rare. Backscattering only induces a partial decreasing of the total displacement. The influence of quenching is therefore smaller in higher dimensions.

As shown in Fig. 3(a), deviations between the quenched and annealed systems occur only after a certain amount of time. If the mean free path is shorter than the average interdisk distance ($\ell_s<\langle\Delta\rangle$), transport in the interdisk region will be diffusive. The step length distribution, as shown in Fig. 4, presents two different behaviors for small and large steps. Small steps ($l<\langle\Delta\rangle$) are determined by a diffusion process, with the step length distribution presenting a relatively fast decay. This part describes 98% of the steps performed in the system. On the other hand, the slowly decaying tail, corresponding to the chord length model for $l>\langle\Delta\rangle$, represents only 2% of the total number of steps. As a result, the initial expansion of the mean-square displacement in the superdiffusive regime is relatively slow. Indeed, at short times ($t<\langle\Delta\rangle/v$), the γ exponent is only $\gamma=1.48$ in both quenched and annealed systems. At later times, for the annealed case, increasingly more random walkers start performing long steps, bringing increasingly larger contributions to the mean-square displacement. The γ exponent then increases, reaching $\gamma=1.71$ in the range $40<t<100$. For the quenched case, a similar acceleration naturally occurs. This acceleration, however, is suppressed to a certain extent by the quenched disorder since the random walker has a nonzero probability to be scattered back into the disk it has just crossed, thereby decreasing its contribution to the mean-square displacement. Due to these step correlations, γ only reaches $\gamma=1.55$ for $40<t<100$, for the case in which quenching is strong.

When the mean free path becomes comparable to the average interdisk distance (e.g., $\ell_s=10\approx\langle\Delta\rangle$), transport in the interdisk region is in a single scattering regime. This results in a faster growth of the mean-square displacement for short times, as well as a much reduced difference between quenched and annealed systems. As the probability to cross the interdisk region and reach another disk is high, a random walker can easily escape from its original neighborhood. This strongly reduces the step length correlations and decreases the influence of quenching. The behavior of such systems is therefore very close to their annealed counterpart, as shown in Fig. 3(a).

B. Comparison with standard Lévy walks

As shown above, the influence of quenched disorder on transport weakens as the ratio between the mean free path and the average interdisk distance increases. For $\ell_s>\langle\Delta\rangle$, the annealed and quenched systems are therefore expected to have similar properties.

Following Eq. (1), a complete description of the step length distribution of a Lévy walk requires the specification of two quantities: the α exponent, which characterizes the asymptotic decay of the step length distribution, and the scale parameter c. This parameter determines the numerical value of the average step length (if $\alpha>1$). Although it does not modify the superdiffusion exponent γ, its value alters the generalized diffusion constant D, as well as the time needed to observe the asymptotic behavior described in Eq. (2).

Figure 5(a) shows the step length distribution of a 2D Lévy glass with $\ell_s=20$ evaluated from Monte Carlo simulations, compared with that of a Lévy walk with $\alpha=1.40$ and scale parameter $c=30$. The asymptotic decays of the two distributions are very similar. The lower value of α compared to the results presented above comes from the fact that large mean free paths ($\ell_s>\langle\Delta\rangle$) imply high probabilities to step across the interdisk region, thereby leading to a slightly accelerated transport. In Fig. 5(b), we compare the mean-square displacement of the Lévy glass with that of the Lévy walk. The maximum diameter for this simulation has been placed at $\phi_n=20\,000$ to avoid all possible effects of the truncation. A very good agreement is found between the two curves, indicating that the transport properties of 2D Lévy glasses with the appropriate low density of scatterers in between the voids can be described to a very good approximation by a standard Lévy walk model (i.e., without step correlations). This result holds for a two-dimensional system and one would expect the effect of quenching to be even smaller in three dimensions.

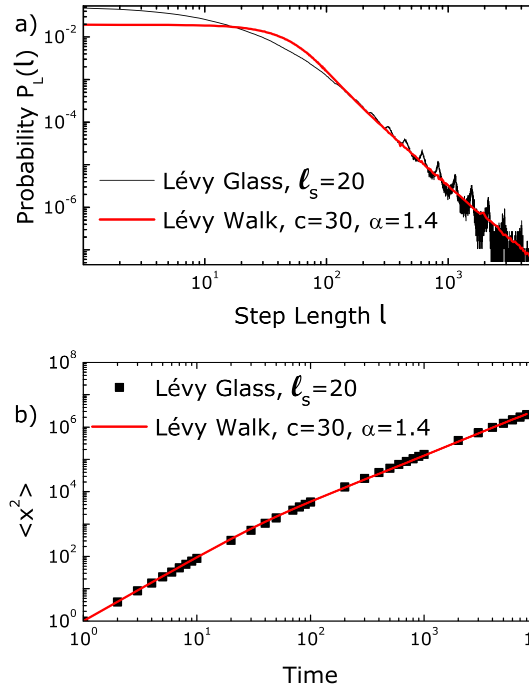

FIG. 5. (Color online) Comparison between a standard Lévy walk model and transport in a 2D Lévy glass with an appropriate scatter density in the region between the voids ($\ell_s=20>\langle\Delta\rangle$), making sure that transport in the region in between the voids is in the single scattering regime. (a) Step length distribution for a Lévy glass (thin black curve) and a Lévy walk (thick red curve). The asymptotic behavior of the step length distribution of the Lévy glass corresponds to a Lévy walk with $\alpha=1.40$. (b) Mean-square displacements for the Lévy glass (black squares) compared to a Lévy walk (red line). The mean-square displacements of the two random walks are similar to each other as quenching effects are negligible.

C. Signature at the truncation time

Up to now, we have considered the transport properties of 2D Lévy glasses at times smaller than the truncation time $t=\phi_n/v$, which is the time needed to cross the largest disk of the sample. While it is well known that transport at larger times becomes unavoidably diffusive owing to the truncated character of the step length distribution, studying how the spreading of random walkers evolves at the transition between the superdiffusive and diffusive regimes may bring valuable information on the system. In the following, we analyze the role played by quenched disorder around the truncation time, when intentionally the strength of quenching is increased by creating a diffusive regime in between the voids.

For these simulations, the maximum diameter was set to $\phi_n=1000$ in order to observe a transition from the initial superdiffusive behavior to the long-time diffusive transport

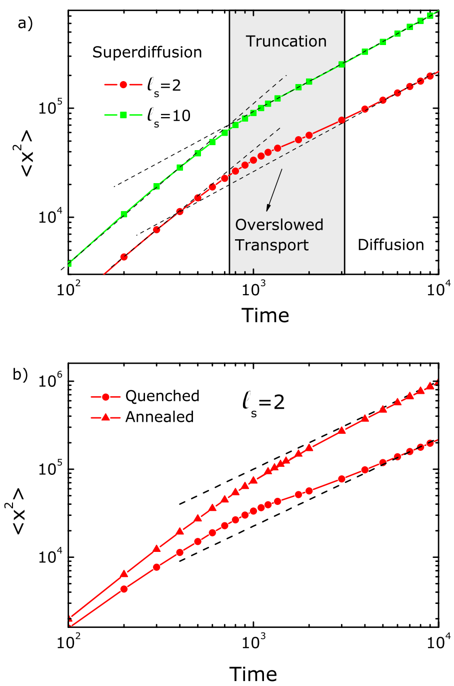

FIG. 6. (Color online) Mean-square displacement at the truncation time in 2D Lévy glasses with maximum diameter $\phi_n = 1000$. (a) Comparison for $\ell_s = 2$ (red dotted line) and $\ell_s = 10$ (green squared line) in the quenched systems. For $\ell_s \approx \langle \Delta \rangle$, the transport regime gradually slows down from the superdiffusive to diffusive. For small mean free paths, the increase in the mean-square displacement close to the truncation is overslowed compared to standard diffusion. Diffusive transport is recovered at very long times. (b) Comparison of quenched (dots) and annealed (triangles) systems for $\ell_s = 2$. The overslowed regime occurs only in the quenched system. The asymptotic diffusion constant of the annealed system ($D = 98.4 \pm 0.2$) is much higher than that of the quenched system ($D = 19.7 \pm 0.1$). Dashed lines: asymptotic diffusive transport.

at times close to $t = 1000$. Figure 6 shows the time evolution of the mean-square displacement close to the truncation time. For large mean free paths ($\ell_s \approx \langle \Delta \rangle$), the transition is smooth and regular. The superdiffusive transport ($\gamma > 1$) gradually slows down to reach a diffusive behavior ($\gamma = 1$). For small mean free paths ($\ell_s < \langle \Delta \rangle$), instead, we find a strong overslowed transport ($\gamma < 1$) close to truncation time. As shown in Fig. 6(b), this effect disappears in the equivalent annealed system, thereby indicating that this subdiffusivelike transport regime is a characteristic of systems with strong quenched disorder.

The transport properties of quenched systems rely on an interplay between propagating and backscattered random walkers. Before the truncation time, many random walkers did not yet finished their steps and, thus, have no backward-propagating counterparts. These walkers bring an important contribution to the superdiffusive transport properties. At very long times, transport is balanced between the two kinds of transport, resulting in the diffusive behavior observed asymptotically. Close to the truncation time, if quenching effects are strong ($\ell_s < \langle \Delta \rangle$), most of the random walkers that have performed the largest steps are backscattered into the same large disk and forced to travel backward. This movement is not compensated by the forward-propagating random walkers. As a result, the transport properties become slower than the asymptotic diffusive behavior and the system presents a transient subdiffusivelike transport regime [$\langle x^2(t) \rangle \sim t^{\gamma}$, with $\gamma < 1$].

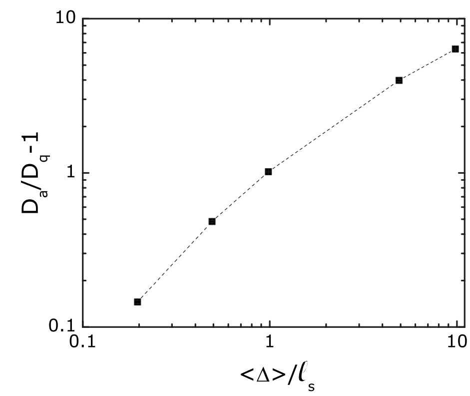

FIG. 7. Strength of quenched disorder, measured by $D_a/D_q - 1$, where D_a and D_q are the asymptotic diffusion constants of the truncated annealed and quenched systems, versus $\langle \Delta \rangle / \ell_s$. The latter parameter constitutes the optical thickness of the material in between the voids. The reduction in D_q compared to D_a is only due to the correlations between consecutive steps. The larger this parameter, the larger the effects of quenching, thus quenched disorder is strong for large $\langle \Delta \rangle / \ell_s$ and becomes negligible when $\langle \Delta \rangle / \ell_s < 1$.

Although for large times the mean-square displacement increases linearly with time, indicating diffusive transport, the effect of quenching on transport is noticed by a reduction in the diffusion constant. For $\ell_s = 2$, the diffusion constant in the annealed case ($D_a = 98.4 \pm 0.2$) is about five times higher than the diffusion constant in the quenched case ($D_q = 19.7 \pm 0.1$). The ratio between the diffusion constants in the annealed and quenched cases decreases with the mean free path ℓ_s. For $\ell_s = 10$, this ratio is $D_a/D_q = 2.02 \pm 0.01$. Comparing the diffusion constant at long times of the quenched and annealed systems makes it possible to estimate the strength of quenched disorder. In annealed systems, the diffusion constant D_a is determined by the first moment of the step length distribution. After the truncation, the influence of quenching is completely contained in the reduction in this asymptotic diffusion constant D_q. In Fig. 7, we present the parameter $D_a/D_q - 1$ as a measure of the strength of the effect of quenching on the transport. We plot this parameter against the optical thickness of the material in between voids, that is, $\langle \Delta \rangle / \ell_s$. We see that $D_a/D_q - 1$ strongly decreases as the optical thickness in the region between the voids decreases and tends to zero for small $\langle \Delta \rangle / \ell_s$. Interestingly, in our system, this parameter is on the order of unity for $\langle \Delta \rangle / \ell_s = 1$.

V. CONCLUSION

In this paper, we have studied the role of quenched disorder on superdiffusive transport in 2D Lévy glasses by means of Monte Carlo simulations. We have shown that the influence of quenched disorder can be controlled by modifying the optical thickness of the region in between the voids.

In the optimal design, the average optical thickness of the material in between the voids should be smaller than 1 or, in other words, the average distance the random walker has to cross in between voids should be smaller than the mean free path in this region ($\langle\Delta\rangle/\ell_s < 1$). This assures that on average a single scattering event takes in between two void crossings. We have shown that in that case the differences between quenched and annealed systems become negligible, and transport is close to an uncorrelated Lévy walk.

The effect of quenching can intentionally be increased, however, by using a higher optical thickness in between the voids. When $\langle\Delta\rangle/\ell_s > 1$, quenching becomes important and leads to interesting effects. When $\langle\Delta\rangle/\ell_s$ is very large, the random walker can be "trapped" in the vicinity of a few large voids for a long time, which slows down superdiffusion. Around the truncation time this can also lead to an interesting transient subdiffusivelike behavior. Finally, we have used the variation in the asymptotic diffusion constant of the quenched and annealed systems to quantify the effective strength of the effect of quenching on transport in Lévy glasses.

The study of transport in Lévy glasses hopefully can contribute to the general understanding of anomalous transport processes in finite-size systems. From an experimental point of view several parameters can be easily tuned, including the superdiffusion exponent. Here, we have shown that one can also tune the effect of quenching on transport in such systems and therefore perform a systematic study of this effect.

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